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Positive groups on $\mathscr{H}^n$ are completely positive

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Abstract

We prove that an operator generates a positive group on the real space of real or complex Hermitian matrices, if and only if it is a Lyapunov operator. In particular it follows that every group of positive operators in fact is a group of completely positive operators.

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1. Notation and problem statement

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ let $\mathscr{H}^n \subset \mathbb{K}^{n\times n}$ denote the real space of $n \times n$ real or complex Hermitian matrices, and $\mathscr{H}^n_+ \subset \mathscr{H}^n$ the set of positive semidefinite matrices. It is well-known that $\mathscr{H}^n$ together with the inner product $\langle X, Y\rangle = \operatorname{trace} XY$ is a Hilbert space and $\mathscr{H}^n_+$ is a self-dual closed normal solid convex cone (e.g. [8]). We write A^* for the conjugate transpose of a matrix $A \in \mathbb{K}^{n\times n}$.

Definition 1.1. A linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ is said to be

(i) *positive*, if $T(H^n_+) \subset \mathscr{H}^n_+$,
(ii) *completely positive*, if T has a representation of the form

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$$T : X \mapsto \sum_{j=1}^{N} A_j X A_j^*$$

with some $N \in \mathbb{N}$ and matrices $A_j \in \mathbb{K}^{n\times n}$,

(iii) *a generator of a (completely) positive group*, if $e^{tT} : \mathscr{H}^n \to \mathscr{H}^n$ is (completely) positive for all $t \in \mathbb{R}$,

(iv) *a Lyapunov operator*, if there exists an $A \in \mathbb{K}^{n\times n}$ such that for all $X \in \mathscr{H}^n$

$$T(X) = AX + XA^*.$$

In this case we write $T = L_A$.

Clearly, every completely positive operator is positive but not vice versa. Counterexamples can be found e.g. in [3,4].

Our principal aim is to prove that Lyapunov operators and generators of positive groups on $\mathscr{H}^n$ are the same.

Theorem 1.2. *A linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ is a generator of a positive group if and only if T is a Lyapunov operator.*

A close relative of this result has been proven by Lindblad [9] (see also [1]). In our terms, it states that a linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ generates a *completely* positive group if and only if T is a Lyapunov operator. Our Theorem 1.2 is therefore equivalent to the assertion in the title of this paper.

2. Exponentially positive operators

Both for the proof and further interpretation of Theorem 1.2, it is useful to recall the following definitions from [2,5,10].

Definition 2.1. A linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ is said to be

(i) *exponentially positive*, if $e^{tT} : \mathscr{H}^n \to \mathscr{H}^n$ is positive for all $t \geqslant 0$.

(ii) *resolvent positive*, if $(\alpha I - T)^{-1}$ is positive for sufficiently large $\alpha > 0$,

(iii) *quasi-monotonic*, if for all $X \in \mathscr{H}^n_+$ there exists a $Y \in \mathscr{H}^n_+$ such that $\langle X, Y\rangle = 0$ and $\langle T(X), Y\rangle \geqslant 0$,

(iv) *cross-positive*, if $\langle X, Y\rangle = 0$ for $X, Y \in H^n_+$ implies $\langle T(X), Y\rangle \geqslant 0$,

(v) *essentially positive*, if $T \in \mathrm{cl}\{S - \alpha I \mid S : \mathscr{H}^n \to \mathscr{H}^n \text{ is positive}\}$.

For general finite-dimensional vector spaces ordered by a closed normal solid cone, it was shown by Elsner in [5] that all these properties are equivalent.

Theorem 2.2. *For a linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ the properties* (i)–(v) *from Definition* 2.1 *are equivalent.*

It is obvious that T generates a positive group on $\mathscr{H}^n$, if and only if both T and $-T$ are exponentially positive. Hence we have the following criterion.

Lemma 2.3. *A linear operator $T : \mathscr{H}^n \to \mathscr{H}^n$ generates a positive group, if and only if $\langle X, Y\rangle = 0$ for $X, Y \in H_+^n$ implies $\langle T(X), Y\rangle = 0$.*

Proof. By Theorem 2.2 the operator $T : \mathscr{H}^n \to \mathscr{H}^n$ generates a positive group, if and only if both T and $-T$ are cross-positive. The latter holds if and only if $\langle X, Y\rangle = 0$ for $X, Y \in H_+^n$ implies both $\langle T(X), Y\rangle \geqslant 0$ and $-\langle T(X), Y\rangle \geqslant 0$. $\square$

Using the concept of resolvent positivity, one easily verifies that all Lyapunov operators generate positive groups.

Lemma 2.4. *Every Lyapunov operator generates a positive group on $\mathscr{H}^n$.*

Proof. Let $T = L_A$ for some $A \in \mathbb{K}^{n\times n}$. By Lyapunov's Theorem (e.g. [7]), L_A^{-1} is positive if $\sigma(L_A) \subset \mathbb{C}_+$. Thus $(\alpha I - L_A)^{-1} = (L_{-A+\frac{\alpha}{2}I})^{-1}$ is positive for sufficiently large $\alpha \in \mathbb{R}$. Hence T is resolvent positive and thus exponentially positive. Since $-T = L_{-A}$ is a Lyapunov operator, too, it is exponentially positive as well. By Lemma 2.3 it follows that T generates a positive group. $\square$

Remark 2.5

(i) It is immediate to see that the set of cross-positive operators $T : \mathscr{H}^n \to \mathscr{H}^n$ is a solid convex cone. Theorem 1.2 characterizes the maximal subspace contained in this cone as the set of Lyapunov operators.

(ii) The role of Lyapunov operators in stability theory is well-known. If for instance we consider a homogeneous linear deterministic differential equation $\dot{X} = AX$, $X(0) = X_0$, with the solution $X(t)$ for $t \in \mathbb{R}$, then $P(t) = X(t)X(t)^*$ satisfies the equation $\dot{P} = L_A(P)$. Since $P(t) \geqslant 0$ for all $t \in \mathbb{R}$, we conclude again that L_A generates a positive group.

(iii) If we consider the stochastic differential equation of Itô type

$$\mathrm{d}X = AX\,\mathrm{d}t + \sum_{j=1}^{N} A_0^{(j)} X\,\mathrm{d}w_j, \quad X(0) = X_0, \tag{1}$$

then for $t \geqslant 0$ the second moment $P(t) = E(X(t)X(t)^*) \geqslant 0$ satisfies

$$\dot{P} = L_A(P) + \sum_{j=1}^{N} A_0^{(j)} P A_0^{(j)*}. \tag{2}$$

Hence, the right-hand side of (2) generates a positive semigroup, but—in general—not a positive group, since (1) cannot be solved for $t < 0$.

Before proceeding with the proof of Theorem 1.2, we verify that the situation is different, if T is a *discrete-time Lyapunov operator* or *Stein operator*. This means that there exists an $A \in \mathbb{K}^{n\times n}$ such that

$$\forall X \in \mathscr{H}^n : \quad T(X) = A^*XA - X. \tag{3}$$

In this case we also write $T = S_A$. It follows from Theorem 2.2(v) that S_A is resolvent positive. But this is not necessarily the case for $T = -S_A$ as the following example shows.

Example 1. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $X_t = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$ for $t > 0$.

For all $\alpha > 0$ we have $\alpha X_t + S_A(X_t) = \begin{bmatrix} t & \alpha - 1 \\ \alpha - 1 & \alpha t \end{bmatrix}$, which is positive for large t though X_t is always indefinite. Hence $(\alpha I + S_A)^{-1}$ is not positive for any α, and hence $-S_A$ is not resolvent positive.

3. Proof of Theorem 1.2

It remains to show that a generator of a positive group on $\mathscr{H}^n$ is necessarily a Lyapunov operator. We distinguish between the real case $\mathbb{K} = \mathbb{R}$ and the complex case $\mathbb{K} = \mathbb{C}$. Though the idea in both cases is basically the same, the complex case is technically more involved and requires some extra considerations.

3.1. The real case

Let T generate a positive group. By Lemma 2.3 this is equivalent to

$$\big(X, Y \geqslant 0 \text{ and } \langle X, Y\rangle = 0\big) \quad \Rightarrow \quad \langle TX, Y\rangle = 0. \tag{4}$$

If e_j denotes the jth canonical unit vector in $\mathbb{R}^n$, then the set

$$B := \left\{e_j e_k^T + e_k e_j^T \mid 1 \leqslant j \leqslant k \leqslant n\right\} \subset \mathscr{H}^n \tag{5}$$

forms a basis of $\mathscr{H}^n \subset \mathbb{R}^{n\times n}$.

It suffices to find an $A \in \mathbb{R}^{n\times n}$ such that $T(X) = L_A(X)$ for all $X \in B$.

Let $X = e_j e_j^T$ (i.e. $2X \in B$). To apply criterion (4) we characterize all matrices

$$Y \in \mathscr{H}^n_+ \quad \text{such that } \langle X, Y\rangle = 0. \tag{6}$$

Let $Y \geqslant 0$ and $\langle X, Y\rangle = y_{jj} = 0$. Then necessarily the jth row and column in Y vanish. Hence, (6) is true if and only if in $Y \geqslant 0$ the jth row and column vanish. Criterion (4) in turn implies that in $T(X)$ everything vanishes except for the jth row and column. Otherwise we could choose some Y satisfying (6) and $\langle T(X), Y\rangle \neq 0$.

For $j = 1, \dots, n$ we thus have $T(e_j e_j^T) = a_j e_j^T + e_j a_j^T$ with vectors $a_1, \dots, a_n \in \mathbb{R}^n$. If we build the matrix $A = (a_1, \dots, a_n)$, then $T(X) = AX + XA^T$ for all $X = e_j e_j^T$. In other words, we have found a unique candidate for the Lyapunov operator.

It remains to show that also for $X_{jk} = e_j e_k^T + e_k e_j^T$ with $j < k$ we have

$$
\begin{aligned}
T(X_{jk}) &= AX_{jk} + X_{jk}A^T = a_j e_k^T + a_k e_j^T + e_j a_j^T + e_k a_k^T \\
&= \begin{bmatrix}
 & & a_{1k} & \cdots & a_{1j} & & \\
 & & \vdots & & \vdots & & \\
a_{1k} & \cdots & 2a_{jk} & \cdots & a_{jj} + a_{kk} & \cdots & a_{nk} \\
 & & \vdots & & \vdots & & \\
a_{1j} & \cdots & a_{jj} + a_{kk} & \cdots & 2a_{kj} & \cdots & a_{nj} \\
 & & \vdots & & \vdots & & \\
 & & a_{nk} & \cdots & a_{nj} & &
\end{bmatrix}.
\end{aligned}
\tag{7}
$$

Let j and k be fixed. A matrix Y satisfies condition (6) with $X = X_{jk} + X_{jj} + X_{kk} \geqslant 0$ if in Y the jth and kth row and column vanish. As above we conclude from criterion (4) that in $T(X)$ and hence also in $T(X_{jk})$ everything vanishes except for the jth and kth row and column. Thus $T(X_{jk})$ is of the general form

$$
\begin{aligned}
T(X_{jk}) &= b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T \quad \text{with } b_j, b_k \in \mathbb{R}^n \\
&= \begin{bmatrix}
 & & b_{1j} & \cdots & b_{1k} & & \\
 & & \vdots & & \vdots & & \\
b_{1j} & \cdots & 2b_{jj} & \cdots & b_{jk} + b_{kj} & \cdots & b_{nj} \\
 & & \vdots & & \vdots & & \\
b_{1k} & \cdots & b_{jk} + b_{kj} & \cdots & 2b_{kk} & \cdots & b_{nk} \\
 & & \vdots & & \vdots & & \\
 & & b_{nj} & \cdots & b_{nk} & &
\end{bmatrix}.
\end{aligned}
\tag{8}
$$

Now we consider matrices of the form $X = xx^T$ with $x = x_j e_j + x_k e_k$ where $x_j, x_k \in \mathbb{R}$ are arbitrary real numbers. Writing

$$X = x_j x_k X_{jk} + x_j^2 e_j e_j^T + x_k^2 e_k e_k^T,$$

and exploiting the linearity of T we have the decomposition

$$
\begin{aligned}
T(X) &= x_j x_k T(X_{jk}) + x_j^2 T\left(e_j e_j^T\right) + x_k^2 T\left(e_k e_k^T\right) \\
&= x_j x_k \left(b_j e_j^T + e_j b_j^T + b_k e_k^T + e_k b_k^T\right) \\
&\quad + x_j^2\left(a_j e_j^T + e_j a_j^T\right) + x_k^2\left(a_k e_k^T + e_k a_k^T\right).
\end{aligned}
\tag{9}
$$

Let $y \perp x$, for instance

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \quad \text{with} \begin{cases} y_j = x_k, \\ y_k = -x_j, \\ y_\ell \text{ arbitrary for } \ell \notin \{j,k\}. \end{cases} \tag{10}$$

Then $Y = yy^T$ satisfies condition (6), and by (4) we have $\langle T(X), Y\rangle = 0$.

If we write $T(X)$ like in Eq. (9) we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\langle T(X), Y\rangle = \frac{1}{2}\operatorname{trace}(T(X)Y) = \frac{1}{2}y^T T(X) y \\ &= x_j x_k \left(y_j \sum_{\ell=1}^n b_{\ell j} y_\ell + y_k \sum_{\ell=1}^n b_{\ell k} y_\ell \right) + x_j^2 y_j \sum_{\ell=1}^n a_{\ell j} y_\ell + x_k^2 y_k \sum_{\ell=1}^n a_{\ell k} y_\ell \\ &= x_j x_k^3 (b_{jj} - a_{jk}) + x_j^3 x_k (b_{kk} - a_{kj}) + x_j^2 x_k^2 (-b_{kj} - b_{jk} + a_{jj} + a_{kk}) \\ &\quad + x_j x_k^2 \sum_{\ell \notin \{j,k\}} y_\ell (b_{\ell j} - a_{\ell k}) + x_j^2 x_k \sum_{\ell \notin \{j,k\}} y_\ell (-b_{\ell k} + a_{\ell j}). \end{aligned}$$

The right hand side is a homogeneous polynomial in the real unknowns x_j, x_k, and y_ℓ for $\ell \notin \{j,k\}$. Since these unknowns can be chosen arbitrarily, all the coefficients of the polynomial necessarily vanish, i.e.

$$\begin{aligned} b_{jj} &= a_{jk}, \\ b_{kk} &= a_{kj}, \\ b_{kj} + b_{jk} &= a_{jj} + a_{kk}, \\ b_{\ell j} &= a_{\ell k}, \\ b_{\ell k} &= a_{\ell j}. \end{aligned}$$

Inserting these data into (8), we see that (7) holds.

This concludes the proof for the real case.

3.2. *The complex case*

In the complex case $\mathbb{K} = \mathbb{C}$ the computations are a little bit more involved, because we have to deal with real and imaginary parts (which are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ for $z \in \mathbb{C}$). It has to be noted that now $\dim \mathscr{H}^n = n^2$ while in the real case the dimension was $n(n+1)/2$. In particular B from (5) must be completed to a basis $B \cup B_\mathrm{i}$ by

$$B_\mathrm{i} = \left\{ \mathrm{i}e_j e_k^* - \mathrm{i}e_k e_j^* \mid 1 \leqslant j < k \leqslant n \right\} \subset \mathscr{H}^n. \tag{11}$$

Like in the real case we obtain a candidate for the Lyapunov operator L_A with $A = (a_1, \ldots, a_n)^* \in \mathbb{C}^{n\times n}$ from the relations $T(e_j e_j^*) = a_j e_j^* + e_j a_j^*$. But this candidate is not unique in the complex case; we can add an arbitrary diagonal matrix with purely imaginary entries to A without changing $(e_j e_j^*)A + A^*(e_j e_j^*)$. In particular, for arbitrarily given real numbers $\beta_1, \ldots, \beta_{n-1} \in \mathbb{R}$ we can choose the a_{jj} such that

$$\operatorname{Im}(a_{jj} - a_{nn}) = \beta_j, \quad j = 1, \ldots, n-1. \tag{12}$$

This will be needed at the end of the proof. For the moment, we choose *some* matrix $A = (a_1, \ldots, a_n)^* \in \mathbb{C}^{n\times n}$ such that $T(e_j e_j^*) = a_j e_j^* + e_j a_j^*$ for $j = 1, \ldots, n$.

For $X_{jk} = e_j e_k^* + e_k e_j^* \in B$ and $X^{\mathrm{i}}_{jk} = \mathrm{i} e_j e_k - \mathrm{i} e_k e_j^* \in B_{\mathrm{i}}$ we have to verify that

$$T(X_{jk}) = AX_{jk} + X_{jk}A^* = a_j e_k^* + a_k e_j^* + e_j a_j^* + e_k a_k^*$$
$$= \begin{bmatrix} & & a_{1k} & \cdots & a_{1j} & & \\ & & \vdots & & \vdots & & \\ \bar{a}_{1k} & \cdots & 2\operatorname{Re} a_{jk} & \cdots & a_{jj} + \bar{a}_{kk} & \cdots & \bar{a}_{nk} \\ & & \vdots & & \vdots & & \\ \bar{a}_{1j} & \cdots & \bar{a}_{jj} + a_{kk} & \cdots & 2\operatorname{Re} a_{kj} & \cdots & \bar{a}_{nj} \\ & & \vdots & & \vdots & & \\ & & a_{nk} & \cdots & a_{nj} & & \end{bmatrix} \tag{13}$$

and

$$T(X^{\mathrm{i}}_{jk}) = AX^{\mathrm{i}}_{jk} + X^{\mathrm{i}}_{jk}A^* = \mathrm{i}a_j e_k^* - \mathrm{i}a_k e_j^* + \mathrm{i}e_j a_k^* - \mathrm{i}e_k a_j^*$$
$$= \begin{bmatrix} & & -\mathrm{i}a_{1k} & \cdots & \mathrm{i}a_{1j} & & \\ & & \vdots & & \vdots & & \\ \mathrm{i}\bar{a}_{1k} & \cdots & -\mathrm{i}a_{jk} + \mathrm{i}\bar{a}_{jk} & \cdots & \mathrm{i}a_{jj} + \mathrm{i}\bar{a}_{kk} & \cdots & \mathrm{i}\bar{a}_{nk} \\ & & \vdots & & \vdots & & \\ -\mathrm{i}\bar{a}_{1j} & \cdots & -\mathrm{i}\bar{a}_{jj} - \mathrm{i}a_{kk} & \cdots & -\mathrm{i}\bar{a}_{kj} + \mathrm{i}a_{kj} & \cdots & \bar{a}_{nj} \\ & & \vdots & & \vdots & & \\ & & -\mathrm{i}a_{nk} & \cdots & \mathrm{i}a_{nj} & & \end{bmatrix}. \tag{14}$$

Let j and k be fixed. Like in the real case we conclude that $T(X_{jk})$ is of the general form

$$T(X_{jk}) = b_j e_j^* + e_j b_j^* + b_k e_k^* + e_k b_k^* \quad \text{with } b_j, b_k \in \mathbb{R}^n$$
$$= \begin{bmatrix} & & b_{1j} & \cdots & b_{1k} & & \\ & & \vdots & & \vdots & & \\ \bar{b}_{1j} & \cdots & 2\operatorname{Re} b_{jj} & \cdots & b_{jk} + \bar{b}_{kj} & \cdots & \bar{b}_{nj} \\ & & \vdots & & \vdots & & \\ \bar{b}_{1k} & \cdots & \bar{b}_{jk} + b_{kj} & \cdots & 2\operatorname{Re} b_{kk} & \cdots & \bar{b}_{nk} \\ & & \vdots & & \vdots & & \\ & & b_{nj} & \cdots & b_{nk} & & \end{bmatrix}.$$

Clearly $T(X^{\mathrm{i}}_{jk})$ has the same form with b_j, b_k replaced by some $c_j, c_k \in \mathbb{C}^n$.

Now we consider matrices of the form $X = xx^*$ with $x = x_j e_j + x_k e_k$ where $x_j, x_k \in \mathbb{C}$ are arbitrary complex numbers. Writing

$$X = \operatorname{Re}(\bar{x}_j x_k) X_{jk} - \operatorname{Im}(\bar{x}_j x_k) X^{\mathrm{i}}_{jk} + |x_j|^2 e_j e_j^* + |x_k|^2 e_k e_k^*,$$

we have the decomposition

$$\begin{aligned} T(X) = {} & \operatorname{Re}(\bar{x}_j x_k)(b_j e_j^* + e_j b_j^* + b_k e_k^* + e_k b_k^*) \\ & - \operatorname{Im}(\bar{x}_j x_k)(c_j e_j^* + e_j c_j^* + c_k e_k^* + e_k c_k^*) \\ & + |x_j|^2 (a_j e_j^* + e_j a_j^*) + |x_k|^2 (a_k e_k^* + e_k a_k^*). \end{aligned} \tag{15}$$

Similarly as in (10) we choose $Y = yy^*$ with

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \begin{cases} y_j = \bar{x}_k, \\ y_k = -\bar{x}_j, \\ y_\ell \text{ arbitrary for } \ell \notin \{j,k\}, \end{cases}$$

such that $y \perp x$. Then

$$\begin{aligned} 0 &= \frac{1}{2}\langle T(X), Y\rangle = \frac{1}{2}\operatorname{trace}(T(X)Y) = \frac{1}{2} y^* T(X) y \\ &= \operatorname{Re}(\bar{x}_j x_k) \operatorname{Re}\left(y_j \sum_{\ell=1}^n b_{\ell j} \bar{y}_\ell + y_k \sum_{\ell=1}^n b_{\ell k} \bar{y}_\ell \right) \\ &\quad - \operatorname{Im}(\bar{x}_j x_k) \operatorname{Re}\left(y_j \sum_{\ell=1}^n c_{\ell j} \bar{y}_\ell + y_k \sum_{\ell=1}^n c_{\ell k} \bar{y}_\ell \right) \\ &\quad + |x_j|^2 \operatorname{Re}\left(y_j \sum_{\ell=1}^n a_{\ell j} \bar{y}_\ell \right) + |x_k|^2 \operatorname{Re}\left(y_k \sum_{\ell=1}^n a_{\ell k} \bar{y}_\ell \right) \\ &= \operatorname{Re}(\bar{x}_j x_k) \operatorname{Re}\Bigg(|x_k|^2 b_{jj} - \bar{x}_k x_j b_{kj} + \bar{x}_k \sum_{\ell \notin \{j,k\}} b_{\ell j} \bar{y}_\ell \\ &\qquad - \bar{x}_j x_k b_{jk} + |x_j|^2 b_{kk} - \bar{x}_j \sum_{\ell \notin \{j,k\}} b_{\ell k} \bar{y}_\ell \Bigg) \\ &\quad - \operatorname{Im}(\bar{x}_j x_k) \operatorname{Re}\Bigg(|x_k|^2 c_{jj} - \bar{x}_k x_j c_{kj} + \bar{x}_k \sum_{\ell \notin \{j,k\}} c_{\ell j} \bar{y}_\ell \\ &\qquad - \bar{x}_j x_k c_{jk} + |x_j|^2 c_{kk} - \bar{x}_j \sum_{\ell \notin \{j,k\}} c_{\ell k} \bar{y}_\ell \Bigg) \end{aligned}$$

$$+|x_j|^2 \operatorname{Re}\left(|x_k|^2 a_{jj} - \bar{x}_k x_j a_{kj} + \bar{x}_k \sum_{\ell \notin \{j,k\}} a_{\ell j}\bar{y}_\ell\right)$$
$$+|x_k|^2 \operatorname{Re}\left(-\bar{x}_j x_k a_{jk} + |x_j|^2 a_{kk} + \bar{x}_j \sum_{\ell \notin \{j,k\}} a_{\ell k}\bar{y}_\ell\right).$$

We distinguish between several special cases, where the variables x_j, x_k and y_ℓ are of the form $x_j = \xi_j z_j$, $x_k = \xi_k z_k$, $y_\ell = \eta_\ell z_\ell$ with fixed complex numbers z_j, z_k, z_ℓ and real variables ξ_j, ξ_k, η_ℓ. In these cases the right hand side is a homogeneous, identically vanishing polynomial in ξ_j, ξ_k, η_ℓ. By inspecting the coefficients at different monomials we obtain relations between the entries of A, b_j, b_k, c_j and c_k. For each case considered, we provide a list of some monomials and the corresponding vanishing coefficients.

(i) Case $x_j = \xi_j$, $x_k = \xi_k$.

	$y_\ell = \eta_\ell$	$y_\ell = \mathrm{i}\eta_\ell$
$\xi_j\xi_k^3$	$\operatorname{Re}(b_{jj} - a_{jk}) = 0$	
$\xi_j^3\xi_k$	$\operatorname{Re}(b_{kk} - a_{kj}) = 0$	
$\xi_j^2\xi_k^2$	$\operatorname{Re}(-b_{kj} - b_{jk} + a_{jj} + a_{kk}) = 0$	
$\xi_j\xi_k^2\eta_\ell$	$\operatorname{Re}(b_{\ell j} - a_{\ell k}) = 0$	$\operatorname{Im}(b_{\ell j} - a_{\ell k}) = 0$
$\xi_j^2\xi_k\eta_\ell$	$\operatorname{Re}(-b_{\ell k} + a_{\ell j}) = 0$	$\operatorname{Im}(-b_{\ell k} + a_{\ell j}) = 0$

Hence b_{jj}, b_{kk} and $b_{\ell j}$, $b_{\ell k}$ for all $\ell \notin \{j, k\}$ have the required form.

(ii) Case $x_j = \xi_j$, $x_k = \mathrm{i}\xi_k$.

	$y_\ell = \eta_\ell$	$y_\ell = \mathrm{i}\eta_\ell$
$\xi_j\xi_k^3$	$\operatorname{Re}(c_{jj}) - \operatorname{Im}(a_{jk}) = 0$	
$\xi_j^3\xi_k$	$\operatorname{Re}(c_{kk}) - \operatorname{Im}(a_{kj}) = 0$	
$\xi_j^2\xi_k^2$	$\operatorname{Im}(c_{jk} - c_{kj}) - \operatorname{Re}(a_{jj} + a_{kk}) = 0$	
$\xi_j\xi_k^2\eta_\ell$	$\operatorname{Re}(c_{\ell j}) - \operatorname{Im}(a_{\ell k}) = 0$	$\operatorname{Im}(c_{\ell j}) - \operatorname{Re}(a_{\ell k}) = 0$
$\xi_j^2\xi_k\eta_\ell$	$\operatorname{Im}(c_{\ell k}) + \operatorname{Re}(a_{\ell j}) = 0$	$\operatorname{Re}(c_{\ell k}) + \operatorname{Im}(a_{\ell j}) = 0$

Hence c_{jj}, c_{kk} and $c_{\ell j}$, $c_{\ell k}$ for all $\ell \notin \{j, k\}$ have the required form.

(iii) Case $x_j = \xi_j$, $x_k = (1 + \mathrm{i})\xi_k$, $y_\ell = 0$.
Considering the coefficient at $\xi_j^2\xi_k^2$ we obtain

$$0 = \operatorname{Re}\big(-(1-\mathrm{i})b_{kj} - (1+\mathrm{i})b_{jk} + 2a_{jj}$$
$$+ (1-\mathrm{i})c_{kj} + (1+\mathrm{i})c_{jk} + 2a_{kk}\big)$$

$$
\begin{aligned}
&= \mathrm{Re}(-b_{kj} - b_{jk} + a_{jj} + a_{kk}) + \mathrm{Im}(-b_{kj} + b_{jk}) \\
&\quad - \mathrm{Re}(-c_{kj} - c_{jk}) - \mathrm{Im}(-c_{kj} + c_{jk}) + \mathrm{Re}(a_{jj} - a_{kk}) \\
&= \mathrm{Im}(-b_{kj} + b_{jk}) - \mathrm{Re}(c_{kj} + c_{jk}).
\end{aligned}
$$

Here we have made use of the corresponding relations in the cases (i) and (ii).

The proof would be complete, if we could show that

$$\mathrm{Im}(-b_{kj} + b_{jk}) = \mathrm{Im}(a_{jj} - a_{kk}) = -\mathrm{Re}(c_{kj} + c_{jk}).$$

But without any further specification of the $\mathrm{Im}\, a_{jj}$ this need not be true. Recall from (12) that there was some freedom in their choice. Notice further that the $\mathrm{Im}\, a_{jj}$ have played no role in our considerations so far.

Hence, to finish the proof, let us consider the difference $S = T - L_A$, which by assumption and Lemma 2.4 also satisfies (4). For $s_{jk} := \mathrm{Im}(-b_{kj} + b_{jk} - a_{jj} + a_{kk})$ we know from the cases (i)–(iii) that

$$S(X_{jk}) = s_{jk} X^{\mathrm{i}}_{jk} \quad \text{and} \quad S(X^{\mathrm{i}}_{jk}) = s_{jk} X_{jk}. \tag{16}$$

Since j and k were arbitrary, we conclude that for all $j < k$ there exist real numbers s_{jk} such that (16) holds. By the construction of A we have $S(X_{jj}) = 0$ for all $j = 1, \ldots, n$. As noted in (12) we can assume that $\mathrm{Im}(a_{jj} - a_{nn}) = \mathrm{Im}(-b_{nj} + b_{jn})$ for $1 \leqslant j < n$, i.e.

$$s_{1n} = s_{2n} = \ldots = s_{n-1,n} = 0.$$

This determines A up to an imaginary multiple of the identity matrix, which can always be added to A without changing L_A.

Now let $j < k < n$ be fixed again and set $x = e_j + \mathrm{i}e_k + e_n$ and $y = e_j + e_k - (1 - \mathrm{i})e_n$ such that $x^*y = 0$. Taking into account that $S(X_{jj}) = S(X_{kk}) = S(X_{nn}) = 0$ and $s_{jn} = s_{kn} = 0$ we find

$$0 = y^* S(xx^*) y = y^*(s_{jk} X_{jk}) y = 2 s_{jk}.$$

Hence $S = 0$ and therefore $T = L_A$.

This concludes the proof for the complex case as well.

4. An open question

Following [6], we use the term *lineality space* for the maximal subspace contained in the cone of resolvent positive operators on some ordered vector space. In [11] the interesting question was posed whether every resolvent positive operator T can be represented as the sum

$$T = L + P \tag{17}$$

of a positive operator P and an element L from the lineality space. Although in [11] an affirmative answer could be given for important classes of cones, it was shown in

[6] that this representation is impossible for almost all cones in a certain categorial sense.

Nevertheless, the question seems to be still open for the ordered vector space of Hermitian matrices. In view of our main result, we ask: Is it true that every exponentially positive operator on $\mathscr{H}^n$ can be decomposed in the sum of a Lyapunov operator and a positive operator?

Again, the analogous result for completely positive operators is already available in [9]: If $T : \mathscr{H}^n \to \mathscr{H}^n$ generates a semigroup of *completely* positive operators, then T possesses a so-called *Lindblad-decomposition* $T = L + P$, where L is a Lyapunov operator and P is *completely* positive.

References

[1] P. Ara, M. Mathieu, Local Multipliers of C^*-Algebras, Springer-Verlag, 2003.

[2] A. Berman, M. Neumann, R.J. Stern, Nonnegative Matrices in Dynamic Systems, John Wiley & Sons, New York, 1989.

[3] S.J. Cho, S.-H. Kye, S.G. Lee, Generalized Choi maps in three-dimensional matrix algebras, Linear Algebra Appl. 171 (1992) 212–224.

[4] M.-D. Choi, Positive semidefinite biquadratic forms, Linear Algebra Appl. 4 (1975) 95–100.

[5] L. Elsner, Quasimonotonie und Ungleichungen in halbgeordneten Räumen, Linear Algebra Appl. 8 (1974) 249–261.

[6] P. Gritzmann, V. Klee, B.-S. Tam, Cross-positive matrices revisited, Linear Algebra Appl. 223/224 (1995) 285–305.

[7] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.

[8] M.A. Krasnosel'skij, J.A. Lifshits, A.V. Sobolev, Positive Linear Systems—The Method of Positive Operators, Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989.

[9] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. 48 (1976) 119–130.

[10] H. Schneider, M. Vidyasagar, Cross-positive matrices, SIAM J. Numer. Anal. 7 (4) (1970) 508–519.

[11] R.J. Stern, H. Wolkowitz, Exponential nonnegativity on the ice-cream cone, SIAM J. Matrix Anal. Appl. 12 (1994) 755–778.